

# Turbulence With Pressure

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## Abstract

We investigate the exact results of the Navier-Stokes equations using the methods developed by Polyakov. It is shown that when the velocity field and the density are not independent, the Burgers equation is obtained leading to exact N-point generating functions of velocity field. Our results show that, the operator product expansion has to be generalized both in the absence and the presence of pressure. We find a method to determine the extra terms in the operator product expansion and derive its coefficients and find the first correction to probability distribution function. In the general case and for small pressure, we solve the problem perturbatively and find the probability distribution function for the Navier-Stokes equation in the mean field approximation.

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## 1- Introduction

A theoretical understanding of turbulence has eluded physicists for a long time. A statistical theory of turbulence has been put forward by Kolmogorov [1], and further developed by others [2-4]. This is to model turbulence using stochastic partial differential equations. In this direction, Polyakov [5] has recently offered a field theoretic method for deriving the probability distribution or density of states in  $(1+1)$ -dimensional in the problem of randomly driven Burgers equation. The importance of the Burgers equation is that, it is the simplest equation that resembles the analytic structure of the N-S equation, at least formally, within the scope of applicability of the Kolmogorov's arguments.

Polyakov formulates a new method for analyzing the inertial range correlation functions based on the two important ingredients in field theory and statistical physics namely the operator product expansion (OPE) and anomalies. He argues that in the limit of high Reynold's number because of existence of singularities at coinciding point, dissipation remains finite and all subleading terms give vanishing contributions in the inertial range. By using the OPE one finds the leading singularities and can show that this approach is self-consistent.

Here we generalize Polyakov's approach to the Burgers turbulence [5] in the presence of the pressure (i.e. the Navier-Stokes equation). We consider two distinct situations, firstly when the velocity field and the density are not independent, and secondly when the gradient of the pressure is small. In the first case we show that by a change of variables one can transform this problem to the Burgers equation and following [6] we find the exact N-point generating function of the velocities. For the second case we calculate the effect of pressure perturbatively and find the asymptotic behavior of probability distribution function in the first approximation.

On the other hand, using the results of [6], we calculate the OPE coefficients proposed by Polyakov, and show that we have to correct the OPE both in the presence and the absence of the pressure. We write the generalized OPE and find the first correction to the PDF which was initially found by Polyakov.

The paper is organised as follows. In section two we show that when the velocity and density are not independent the governing equation reduces to the Burgers equation. In section three we find the N-point generating functions for the velocity field and show that we have to generalize the OPE proposed by Polyakov. We find the OPE coefficients and derive the first correction to probability distribution function. In section four we consider the effect of pressure perturbatively and find the probability distribution function in the mean field approximation.

## 2- The Compressible fluid and the Burgers equation

We begin our discussion with the system describing compressible flows in an ideal (polytropic) gas ignoring dissipative effect [7], viz.

$$\rho_t + v\rho_x + \rho v_x = 0 \tag{1}$$

$$\rho(v_t + vv_x) + p_x = 0 \tag{2}$$

$$p = k\rho^\gamma, \quad s = \text{const.} \tag{3}$$

Here  $\rho$ ,  $v$  and  $p$  are the density, velocity field and pressure, respectively. The  $s$  is entropy, assumed to be constant,  $\gamma = \frac{C_p}{C_v}$ , the ratio of specific heats and  $k$  is a constant.

Also in inertial range, we ignore the driving force, which pumps energy in the large scale of system. The system of eqs.(1-3) has essentially two variables ( $\rho$  and  $v$ ), it is nonlinear and difficult to handle in complete generality. Nevertheless there has been considerable analytic

interest in this system. In particular, some progress can be made by seeking simple wave solutions where one of the dependent variables is a functions of the others. Rewriting eq.(1-3) in terms of  $v$  and  $\rho$  only by introducing the square of the speed of sound,  $a^2 = (\frac{\partial p}{\partial \rho})_{s=s_0} = k\gamma\rho^{\gamma-1}$ , we have:

$$\rho_t + v\rho_x + \rho v_x = 0 \quad (4)$$

$$v_t + vv_x + \frac{a^2(\rho)}{\rho}\rho_x = 0 \quad (5)$$

Now, we assume that  $v = v(\rho)$ , so that eqs.(4) and (5) changes to the following equations:

$$\rho_t + (v + \rho v')\rho_x = 0 \quad (6)$$

$$\rho_t + (v + \frac{a^2}{\rho v'})\rho_x = 0 \quad (7)$$

Here the prime denotes differentiation with respect to  $\rho$ . This system of linear algebraic equations in  $\rho_t$  and  $\rho_x$  has a non-trivial solution provided the determinant of the coefficient matrix vanishes so that:

$$v' = \pm \frac{a}{\rho} = \pm a_0 \left(\frac{\rho}{\rho_0}\right)^{\frac{\gamma-1}{2}} \frac{1}{\rho} \quad (8)$$

The system (6-7), then reduces to one of the equations:

$$\rho_t + (v \pm a)\rho_x = 0 \quad (9)$$

where

$$v(\rho) = \int_{\rho_0}^{\rho} \frac{a(\rho)}{\rho} d\rho = \frac{2}{\gamma-1} \{a(\rho) - a_0\} \quad (10)$$

We restrict our attention to waves moving to the right thus choosing the plus sign in eqs.(8) and (9). The corresponding equation for  $v$  follows easily from multiplication of eq.(6) by

$v(\rho)$  and writing the result in terms of  $v$  via eq.(10). Therefore we obtain:

$$v_t + (a_0 + \frac{\gamma + 1}{2}v)v_x = 0 \quad (11)$$

Cole generalizes the eq.(11) in the presence of small kinematical viscosity and have found the following equation for  $v$  [8]:

$$v_t + (a_0 + \frac{\gamma + 1}{2}v)v_x = \nu v_{xx} \quad (12)$$

This equation can be transformed into the Burgers equation by simple change of variables.

We define  $u = a_0 + \frac{\gamma+1}{2}v$  and in the presence of some driving force in large scale eq.(12) transformed to:

$$u_t + uu_x = \nu u_{xx} + \bar{f}(x, t) \quad (13)$$

where  $\bar{f}(x, t)$  is a Gaussian random force with the following correlation:

$$\langle \bar{f}(x, t) \bar{f}(x', t') \rangle = (\frac{\gamma + 1}{2})^2 k(x - x') \delta(t - t') \quad (14)$$

The transformation,  $u(x, t) = -\lambda \partial_x h(x, t)$  maps eq.(13) to the well known Kardar-Parisi-Zhang equation [9],

$$\partial_t h = \nu \partial_{xx} h + \frac{\lambda}{2} [\partial_x h]^2 + \bar{f}_2(x, t) \quad (15)$$

which gives the general features of many complex system [9,10]. It is noted that the parameter  $\lambda$  that appears in the above transformation is not renormalized under any renormalization procedure [11].

### 3- The Exact N-Point Generating Functions and the Generalized OPE

To statistical description of eq.(13) following Polyakov [5] consider the following generating functional

$$Z_N(\lambda_1, \lambda_2, \dots, \lambda_N, x_1, \dots, x_N) = \langle \exp(\sum_{j=1}^N \lambda_j u(x_j, t)) \rangle \quad (16)$$

Noting that the random force  $\bar{f}(x, t)$  has a Gaussian distribution,  $Z_N$  satisfies a closed differential equation provided that the viscosity  $\nu$  tends to zero:

$$\dot{Z}_N + \sum \lambda_j \frac{\partial}{\partial \lambda_j} \left( \frac{1}{\lambda_j} \frac{\partial Z_N}{\partial x_j} \right) = \sum \bar{k}(x_i - x_j) \lambda_i \lambda_j Z_N + D_N \quad (17)$$

where  $D_N$  is :

$$D_N = \nu \sum \lambda_j \langle u''(x_j, t) \exp \sum \lambda_k u(x_k, t) \rangle \quad (18)$$

and

$$\bar{k}(x_i - x_j) = \left( \frac{\gamma + 1}{2} \right)^2 k(x_i - x_j) \quad (19)$$

To remain in the inertial range we must, however, keep  $\nu$  infinitesimal but non-zero. Polyakov argues that the anomaly mechanism implies that infinitesimal viscosity produces a finite effect. To compute this effect Polyakov makes the F-conjecture, which is the existence of an operator product expansion or the fusion rules. The fusion rule is the statement concerning the behaviour of correlation functions, when some subset of points are put close together.

Let us use the following notation;

$$Z(\lambda_1, \lambda_2, \dots, x_1, \dots, x_N) = \langle e_{\lambda_1}(x_1) \dots e_{\lambda_N}(x_N) \rangle \quad (20)$$

then Polyakov's F-conjecture is that in this case the OPE has the following form,

$$e_{\lambda_1}(x + y/2) e_{\lambda_2}(x - y/2) = A(\lambda_1, \lambda_2, y) e_{\lambda_1 + \lambda_2}(x) + B(\lambda_1, \lambda_2, y) \frac{\partial}{\partial x} e_{\lambda_1 + \lambda_2} + o(y^2) \quad (21)$$

This implies that  $Z_N$  fuses into functions  $Z_{N-1}$  as we fuse a couple of points together. The F-conjecture allows us to evaluate the following anomaly operator (i.e. the  $D_N$ -term in eq.(5)),

$$a_\lambda(x) = \lim_{\nu \rightarrow 0} \nu (\lambda u''(x) \exp(\lambda u(x))) \quad (22)$$

which can be written as:

$$a_\lambda(x) = \lim_{\xi, y, \nu \rightarrow 0} \lambda \nu \frac{\partial^3}{\partial \xi \partial y^2} e_\xi(x+y) e_\lambda(x) \quad (23)$$

As discussed in [3] the possible Galilean invariant expression is:

$$a_\lambda(x) = a(\lambda) e_\lambda(x) + \tilde{\beta}(\lambda) \frac{\partial}{\partial x} e_\lambda(x) \quad (24)$$

Therefore in steady state the master equation takes the following form,

$$\begin{aligned} \sum \left( \frac{\partial}{\partial \lambda_j} - \beta(\lambda_j) \right) \frac{\partial}{\partial x_j} Z_N - \sum \bar{k}(x_i - x_j) \lambda_i \lambda_j Z_N &= \sum a(\lambda_j) Z_N \\ \beta(\lambda) &= \tilde{\beta}(\lambda) + \frac{1}{\lambda} \end{aligned} \quad (25)$$

Let us consider following correlation for  $\bar{f}$  in  $k$ -space as follows [12]:

$$\langle \bar{f}(k, t) \bar{f}(k', t') \rangle = \frac{L}{2\pi} \bar{k}_0 \left( \frac{\gamma+1}{2} \right)^2 \delta(k^2 - \frac{1}{L^2}) \delta(k+k') \delta(t-t') \quad (26)$$

therefore we obtain:

$$\langle \bar{f}(x, t) \bar{f}(x', t') \rangle = \bar{k}_0 \left( \frac{\gamma+1}{2} \right)^2 \cos\left(\frac{x-x'}{L}\right) \delta(t-t') \quad (27)$$

In the inertial range where  $x_i - x_j \ll L$ , we can expand the r.h.s. of eq.(27) and find

$$\bar{K}(x_i - x_j) = \bar{k}_0 \left( \frac{\gamma+1}{2} \right)^2 \left( 1 - \frac{(x_i - x_j)^2}{2L^2} \right) \quad (28)$$

Therefore we find the following explicit form of  $Z_2$  for  $\bar{k}(x_i - x_j)$  given by eq.(28):

$$Z_2(\mu y) = e^{\frac{\gamma+1}{3} (\mu y)^{3/2}} \quad (29)$$

and the following expression for density of states as the Laplace transform of  $Z_2$  ;

$$W(u, y) = \int_{c-i\infty}^{c+i\infty} \frac{d\mu}{2\pi i} e^{-\mu u} Z_2(\mu y) \quad (30)$$

where

$$\mu = 2(\lambda_1 - \lambda_2), \quad y = x_1 - x_2.$$

It can be easily shown that with the following definition of variables Polyakov's master equation (i.e. eq.(25) with the scaling conjecture[5] is:

$$\left\{ \frac{\partial^2}{\partial \mu_2 \partial y_2} + \frac{\partial^2}{\partial \mu_3 \partial y_3} - \left( \frac{\gamma+1}{2} \right)^2 (y_2 \mu_2 + y_3 \mu_3)^2 \right\} f_3 = 0 \quad (31)$$

where

$$f_3 = (\lambda_1 \lambda_2 \lambda_3)^{-b_3} Z_3$$

$$\begin{aligned} y_1 &= \frac{x_1 + x_2 + x_3}{3} & y_2 &= x_1 - \frac{x_2 + x_3}{2} & y_3 &= x_2 - x_3 \\ \mu_1 &= \frac{\lambda_1 + \lambda_2 + \lambda_3}{3} & \mu_2 &= \frac{3}{2} \left( \lambda_1 - \frac{\lambda_2 + \lambda_3}{2} \right) & \mu_3 &= 2(\lambda_2 - \lambda_3) \end{aligned} \quad (32)$$

Now we set  $f_3$  as;

$$f_3 = \mu_2^{S_2} \mu_3^{S_3} g_3(\mu_2 y_2, \mu_3 y_3) \quad (33)$$

inserting in eq.(15) results in

$$g_3(\mu_2 y_2, \mu_3, y_3) = e^{\frac{\gamma+1}{3}(\mu_2 y_2 + \mu_3 y_3)^{3/2}} \quad (34)$$

and

$$S_2 = S_3 = -5/4$$

Now if we use the following transformation;

$$y_1 = \frac{x_1 + x_2 + x_3 + \dots x_N}{N}$$



$$\begin{aligned}
y_2 &= x_1 - \frac{x_2 + x_3 + \dots x_N}{N-1} \\
y_3 &= x_2 - \frac{x_3 + x_4 + \dots x_N}{N-2} \\
\text{and} \quad y_N &= x_{N-1} - x_N
\end{aligned} \tag{35}$$

and

$$\begin{aligned}
\mu_1 &= \frac{\lambda_1 + \lambda_2 + \dots + \lambda_N}{N} \\
\mu_2 &= \frac{N}{N-1} \left[ \lambda_1 - \frac{\lambda_2 + \lambda_3 + \dots + \lambda_N}{N-1} \right] \\
\mu_3 &= \frac{N-1}{N-2} \left[ \lambda_2 - \frac{\lambda_3 + \lambda_4 + \dots + \lambda_N}{N-2} \right] \\
\text{and} \quad \mu_N &= 2(\lambda_{N-1} - \lambda_N)
\end{aligned} \tag{36}$$

we get the following partial differential equation for  $f_N$ ;

$$\left\{ \frac{\partial^2}{\partial y_2 \partial \mu_2} + \dots + \frac{\partial^2}{\partial y_N \partial \mu_N} \right\} f_N - \left( \frac{\gamma+1}{2} \right)^2 (y_2 \mu_2 + \dots + y_N \mu_N)^2 f_N = 0 \tag{37}$$

which is solved by:

$$f_N = (\mu_2 \mu_3 \dots \mu_N)^{-[\frac{2N-1}{2(N-1)}]} \exp^{\frac{\gamma+1}{3} (\mu_2 y_2 + \dots + \mu_N y_N)^{3/2}} \tag{38}$$

The parameter  $b_n$  for the  $N$ -point generating functions can be determined by the requirement that  $Z_N$  must be finite in the limit  $\lambda_i \rightarrow 0$ . Therefore we find:

$$b_N = \frac{2N-1}{2N} \tag{39}$$

It follows that the behavior of  $N$ -point correlation functions of  $v$  is:

$$G^{(N)}(x_1, \dots, x_N) \sim \lim_{\lambda \rightarrow 0} \lambda^{-N} \sum_{k=0}^N a_k^{(N)} (\lambda x)^{\frac{3k}{2}} \tag{40}$$

where  $a_k^{(N)}$  are some constants.

Now we try to find the OPE coefficients in eq.(21).

This can be done by using the  $Z_3$  and put  $x_2$  and  $x_3$  close together. Here we calculate the OPE coefficients for the case  $\gamma = 1$  i.e. the ordinary Burgers turbulence and for arbitrary  $\gamma$  we need only small modification.  $Z_3$  and  $Z_2$  are given by eqs.(32) and (29). In eq.(32) we take  $x_3 = x_2 - 2\epsilon$  and it is easy to show that in the limit  $\epsilon \rightarrow 0$  we find:

$$Z_3 \Rightarrow (\lambda_1 \lambda_2 \lambda_3)^{5/6} (\mu_2 \mu_3)^{-5/4} e^{2/3(\mu_2 y_2 + \mu_3 y_3)^{3/2} (1 - \frac{1}{8} \frac{2\lambda_1 - 5\lambda_2 - 5\lambda_3}{\lambda_1 - \lambda_2 - \lambda_3} + \frac{1}{8} \frac{6\lambda_1 + 13\lambda_2 - 19\lambda_3}{(\lambda_1 - \lambda_2 - \lambda_3)(x_1 - x_2)} \epsilon)^{3/2}} \quad (41)$$

where  $\lambda_3 = -(\lambda_1 + \lambda_2)$ . Now we can expand the exponent of eq.(41) in terms of  $\epsilon$ , therefore we find:

$$\begin{aligned} Z_3 &= (\lambda_1 \lambda_2 \lambda_3)^{5/6} (\mu_2 \mu_3)^{-5/4} e^{2/3(\mu_2 y_2 + \mu_3 y_3)^{3/2}} \left(1 - \frac{1}{4}(2\lambda_1 - 5\lambda_2 - 5\lambda_3) \right. \\ &\quad \left( -2(\lambda_1 - \lambda_2 - \lambda_3)^{1/2} (x_1 - x_2)^{3/2} \frac{1}{4}(6\lambda_1 + 13\lambda_2 - 19\lambda_3) \right. \\ &\quad \left. \left. \left( -2(\lambda_1 - \lambda_2 - \lambda_3)^{1/2} (x_1 - x_2)^{1/2} \epsilon + O(\epsilon^2) \right) \right) \right) \end{aligned} \quad (42)$$

Comparing with eq.(21) we find that:

$$A(\lambda_1, \lambda_2, \epsilon) = (-\lambda_1 \lambda_2 (\lambda_1 + \lambda_2))^{5/6} \left(\frac{9}{2} \lambda_1 (\lambda_1 + 2\lambda_2)\right)^{-\frac{5}{6}} \quad (43)$$

$$B(\lambda_1, \lambda_2, \epsilon) = -\frac{A(\lambda_1, \lambda_2, \epsilon)}{16} \left(\frac{25\lambda_1 + 32\lambda_2}{\lambda_1}\right) \epsilon \quad (44)$$

$$C(\lambda_1, \lambda_2, \epsilon) = \frac{7}{8} A(\lambda_1, \lambda_2, \epsilon) \quad (45)$$

Where  $C(\lambda_1, \lambda_2, \epsilon)$  can be treated as the third terms in the OPE of eq.(21) and are given in the following modified OPE:

$$\begin{aligned} e_{\lambda_1}(x + \epsilon/2) e_{\lambda_2}(x - \epsilon/2) &= A(\lambda_1, \lambda_2, \epsilon) e_{\lambda_1 + \lambda_2}(x) + B(\lambda_1, \lambda_2, \epsilon) \frac{\partial}{\partial x} e_{\lambda_1 + \lambda_2} \\ &\quad + C(\lambda_1, \lambda_2, \epsilon) \frac{\partial}{\partial \lambda} e_{\lambda_1 + \lambda_2} + O(\epsilon^2) \end{aligned} \quad (46)$$

Now one can show that the modified OPE leads to following equations for  $a_\lambda(x)$  and  $Z_2$ :

$$a_\lambda(x) = a(\lambda)e_\lambda(x) + \tilde{\beta}(\lambda)\frac{\partial}{\partial x}e_\lambda(x) + \tilde{\gamma}(\lambda)\frac{\partial}{\partial \lambda}e_\lambda(x) + \dots \quad (47)$$

$$(\partial_\mu - \frac{2b}{\mu})\partial_y Z_2 + c\mu\partial_\mu Z_2 - \mu^2 y^2 Z_2 = 0 \quad (48)$$

where we have used  $\tilde{\beta}(\lambda) = \frac{b-1}{\lambda}$ ,  $a = 0$  according [4], and scaling arguments show that  $\tilde{\gamma}(\lambda)$  has the following form:

$$\tilde{\gamma}(\lambda) = c\lambda \quad (49)$$

The eq.(49) have the following asymptotic behavior:

$$Z_2(\mu y) = e^{\frac{\gamma+1}{3}(\mu y)^{3/2} - \frac{\varepsilon}{2}(\mu y)} \quad (50)$$

It appers that in the limit  $c \rightarrow 0$ , we find Polyakov's result for  $Z_2$ . Indeed our results shows that we can find the exact  $Z_2$  for the Burgers equation by calculating the exact form of OPE, proposed by Polyakov by methods disscussed above.

#### 4- Perturbative Calculations of the Pressure

To consider the general situation we have the following set of equations describing the compressible fluid:

$$\rho_t + v\partial_x \rho + \rho\partial_x v = 0 \quad (51)$$

$$v_t + v\partial_x v + \frac{\partial_x p}{\rho} = \nu\partial_x^2 v + f(x, t) \quad (52)$$

$$p = k\rho^\gamma \quad (53)$$

By using the eq.(53) we find:

$$\rho_t + v\partial_x \rho + \rho\partial_x v = 0 \quad (54)$$

$$v_t + v\partial_x v + k\gamma\rho^{\gamma-2}\partial_x\rho = \nu\partial_x^2 v + f(x, t) \quad (55)$$

where the  $\nu$  is the viscosity. We rewrite the eqs.(54) and (55) for different points  $x_1$  and  $x_2$  and multiplying the resulting equations by  $\rho(x_2)$ ,  $\lambda_1\rho(x_2)$ ,  $\rho(x_1)$  and  $\lambda_2\rho(x_1)$  respectively, and all of them to

$$e^{\lambda_1 v(x_1, t) + \lambda_2 v(x_2, t)} \quad (56)$$

and averaging with respect to the external force we find:

$$\begin{aligned} \sum_{j=1}^2 \left( \frac{\partial}{\partial \lambda_j} - \beta(\lambda_j) \right) \frac{\partial}{\partial x_j} Z_2 - \sum_{j=1}^2 \bar{k}(x_i - x_j) \lambda_i \lambda_j Z_2 + F_2 &= \sum_{j=1}^2 a(\lambda_j) Z_2 + \Lambda_2 Z_2 \\ \beta(\lambda) &= \tilde{\beta}(\lambda) + \frac{1}{\lambda} \end{aligned} \quad (57)$$

where

$$Z(\lambda_1, \lambda_2, x_1, x_2) = \langle \rho(x_1) \rho(x_2) e^{\lambda_1 v(x_1) + \lambda_2 v(x_2)} \rangle \quad (58)$$

$$F_2 = \langle k\gamma\rho(x_1)\rho(x_2)e^{\lambda_1 v(x_1) + \lambda_2 v(x_2)} [\lambda_1 \rho^{\gamma-2}(x_1) \partial_{x_1} \rho(x_1) + \lambda_2 \rho^{\gamma-2}(x_2) \partial_{x_2} \rho(x_2)] \rangle \quad (59)$$

and

$$\Lambda_2 = \text{mean} \left( \sum_{i=1}^2 \tilde{\beta}(\lambda_i) \frac{1}{\rho(x_i)} \partial_{x_i} \rho(x_i) \right) \quad (60)$$

For small pressure we can approximate the eq.(59) in the mean field approximation as:

$$F = -\eta Z, \quad \eta = \text{mean}(-k\gamma[\lambda_1 \rho^{\gamma-2}(x_1) \partial_{x_1} \rho(x_1) + \lambda_2 \rho^{\gamma-2}(x_2) \partial_{x_2} \rho(x_2)]) \quad (61)$$

By using  $\beta(\lambda) = \frac{b}{\lambda}$  and  $a(\lambda) = 0$  we find the following asmyptotic expression for  $Z_2$ :

$$Z_2 = e^{\frac{2}{3}(\mu y)^{3/2} - (\Lambda_2 + \eta)(\mu y)^{-1/2}} \quad \mu y \rightarrow \infty \quad (62)$$

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